$$c = x(T) = \int_{t_0}^{T-h_k} \sum_{i=0}^{k} \left[Y(t+h_i, T) B_i(t+h_i) \right] u(t)\, dt + \sum_{i=0}^{k-1} \int_{T-h_{i+1}}^{T-h_i} \left[\sum_{j=0}^{i} Y(t+h_j, T) B_j(t+h_j) \right] u(t)\, dt$$

and so

$$c^T c = \int_{t_0}^{T-h_k} \left[c^T \sum_{i=0}^{k} Y(t+h_i, T) B_i(t+h_i) \right] u(t)\, dt + \sum_{i=0}^{k-1} \int_{T-h_{i+1}}^{T-h_i} \left[c^T \sum_{j=0}^{i} Y(t+h_j, T) B_j(t+h_j) \right] u(t)\, dt = 0.$$

This is a contradiction for $c \neq 0$. Hence, rank $M = n$.

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Observability of a Class of Hyperbolic Distributed Parameter Systems

T. K. YU AND J. H. SEINFELD

Abstract—**Necessary and sufficient conditions are presented for the observability of a class of linear hyperbolic distributed parameter systems. Observations are assumed to be made along paths intersecting the characteristic curves of the system.**

In this correspondence we present some new results on the observability of a class of linear hyperbolic distributed parameter systems. The results are embodied in necessary and sufficient conditions for the recovery of the state of this class of systems. In order to develop the conditions, we present first an extension of the well-known conditions[1] for the observability of linear lumped parameter systems.

Consider the class of systems described by

$$\dot{u}(t) = F(t)u(t), \qquad t \in [0,T] \tag{1}$$

where $u(t)$ is an $(n \times 1)$ state vector and $F(t)$ is an $(n \times n)$ matrix. Let $\Phi(t,0)$ be the fundamental matrix for system (1). Suppose observations of this system are made at h discrete times in the form

$$y_i = H_i u(t_i), \qquad i = 1,2,\cdots,h \tag{2}$$

where the y_i are $(m_i \times 1)$ vectors of observations and the H_i are $(m_i \times n)$ constant matrices. Therefore, the dimension of y_i and of H_i can change at different measurement times. This system is said to be *observable* if we can recover the initial state $u(0)$ from y_i, $i = 1,2,\ldots,h$. We present the following theorem.

Theorem 1: A necessary and sufficient condition for the observability of (1) is that the $\left(n \times \sum_1^h m_i\right)$ matrix, $M = [\Phi(t_1,0)^T H_1^T, \cdots, \Phi(t_h,0)^T H_h^T]$, has rank n.

Proof: Let $r = \sum_1^h m_i$ and v_i denote the ith column vector of M. Also, let $Y = [y_1^T, \cdots, y_h^T]^T = [Y_1, Y_2, \cdots, Y_r]^T$. Form the $(n \times n)$ matrix Q from any n columns of M, $Q = [v_{i_1} v_{i_2} \cdots, v_{i_n}]$. Then there is an $(n \times 1)$ vector $g = [Y_{i_1}, \cdots, Y_{i_n}]^T$ such that $g = Q^T u(0)$. For any $t \in [0,T]$, we can write $g = Q^T \Phi(t,0)^{-1} u(t)$. Hence, to recover $u(t)$ for all $t \in [0,T]$, it is necessary and sufficient that there exists a matrix Q with rank n or, equivalently, M must have rank n.

Example 1: Let us apply this theorem to system (1) with

$$F(t) = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}. \tag{3}$$

Suppose only u_1 can be measured. Say we measure u_1 at two times, t_1 and t_2. Thus, $H_1 = H_2 = [1\ 0]$ (i.e., $n = 2$, $m_1 = 1$, $m_2 = 1$, $r = 2$) and

$$M = \begin{bmatrix} \cos(t_1^2/2) & \cos(t_2^2/2) \\ \sin(t_1^2/2) & \sin(t_2^2/2) \end{bmatrix}. \tag{4}$$

Let t_1 be chosen arbitrarily. Then, in order that M may have rank 2, it is necessary that $t_2 \neq (t_1^2 + 2k\pi)^{1/2}$, $k = 1,2,3\cdots$, for the system to be observable.

OBSERVABILITY OF A CLASS OF HYPERBOLIC SYSTEMS

Consider the class of linear hyperbolic systems governed by

$$\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} = Au, \qquad x \in [0,1], \qquad t \geqslant 0 \tag{5}$$

where u is an $(n \times 1)$ state vector, β is a positive scalar constant, and A is an $(n \times n)$ constant matrix. The system (5) has only one type of characteristic line, namely, $dt/dx = 1/\beta$. Let $\Phi(\tau,0)$ be the $(n \times n)$ fundamental matrix of the system, $\dot{w} = Aw$, $\tau \in [0,1]$. Also, let R_α denote the closed region in the (x,t) plane bounded by $x = 0$, $x = 1$, $t = 0$, and $t = \beta^{-1}x + \alpha$, $\alpha \geqslant 0$.

Definition: The ith observation path $z_i(t)$ is a line in the (x,t) plane with the following properties: $z_i(0) = 1$, $z_i(t)$ crosses each characteristic line in R_α once, and $z_i(t)$ terminates at a point on the characteristic line $t = \beta^{-1}x + \alpha$.

Let there be h distinct observation paths, $z_i(t)$, $i = 1,2,\cdots,h$, that is, $z_i(t)$ and $z_j(t)$ have no common points for $i \neq j$ and $t > 0$. We will denote the value of the state along the ith observation path as $u_{z_i}(t)$. We assume that the observations $y_i(t)$ of the system (5) are made continuously along the h observation paths in the form

$$y_i(t) = H_i u_{z_i}(t), \qquad i = 1,2,\cdots,h, \quad t > 0 \tag{6}$$

where the H_i are constant $(m_i \times n)$ matrices. The observation at the point $(x = 1, t = 0)$ is in the form $y_i(0) = H_0 u_{z_i}(0)$, $i = 1,2,\cdots,h$, where H_0 is a constant $(n \times n)$ matrix. We will call the system of (5) and (6) observable in R_α if $u(x,t)$ in R_α can be recovered from $y_i(t)$.

Theorem 2: A sufficient condition for the observability of (5) and (6) in R_α is: 1) if for any $\tau_1 \neq \tau_2 \neq \cdots \neq \tau_h$, $0 \leq \tau_i \leq 1$, the $(n \times \sum_1^h m_i)$ matrix $L = [\Phi(\tau_1,0)^T H_1^T, \cdots, \Phi(\tau_h,0)^T H_h^T]$ has rank n and 2) H_0 is nonsingular.

Proof: The proof follows directly from that for Theorem 1.

Example 2: Consider the system (5) with $n = 2$, $\beta = 1$, and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}. \tag{7}$$

Let $\alpha = 1/2$. Assume there are two paths in the (x,t) plane along which observations are made:

$$z_1(t) = \begin{cases} 1 - t, & 0 \leq t \leq 1/2 \\ 1/2, & 1/2 \leq t \leq 1 \end{cases} \tag{8}$$

$$z_2(t) = 1, \qquad 0 \leq t \leq 1.5. \tag{9}$$

We choose the following observation matrices: $H_0 = I$, $H_1 = [1\ 0]$, and $H_2 = [0\ 1]$. Thus, at $x = 1$ and $t = 0$, both states are measured, but along $z_1(t)$ only u_1 is measured and along $z_2(t)$ only u_2 is measured. It is straightforward to show that

$$L = \begin{bmatrix} 1 & 0 \\ 1 - e^{-\tau_1} & e^{-\tau_2} \end{bmatrix} \tag{10}$$

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has rank 2 for any $\tau_1 \neq \tau_2$, $0 \leq \tau_1, \tau_2 \leq 1$; hence, the system is observable.

Finally, we consider another class of linear hyperbolic systems, namely, those governed by

$$\frac{\partial u}{\partial t} + \Lambda_1 \frac{\partial u}{\partial x} = \Lambda_2 u, \qquad x \in [0,1], t \geqslant 0 \tag{11}$$

where Λ_1 and Λ_2 are $(n \times n)$ diagonal matrices with diagonal elements $\lambda_{11}, \cdots, \lambda_{1n}$ and $\lambda_{21}, \cdots, \lambda_{2n}$, respectively. Therefore, (11) represents n uncoupled hyperbolic systems, each having its own characteristic line, with slope $dt/dx = \lambda_{1i}^{-1}$, $i = 1,2, \cdots, n$. We assume $\lambda_{11} > \lambda_{12} > \cdots > \lambda_{1n}$.

We define $R_\alpha(i)$ as the closed region on the (x,t) plane bounded by $x = 0$, $x = 1$, $t = 0$, and $t = \lambda_{1i}^{-1}x + \alpha$, $\alpha \geq 0$. We also define the single observation path $z^+(t)$ by: 1) $z^+(0) = 1$; 2) $z^+(t)$ crosses each characteristic line of type i in $R_\alpha(i)$ once, $i = 1,2, \cdots, n$; and 3) $z^+(t)$ terminates at a point on the characteristic line $t = \lambda_{1n}^{-1}x + \alpha$. $z_1^+(t)$ will denote the segment of $z^+(t)$ that begins at the point $(x = 1, t = 0)$ and terminates on the line $t = \lambda_{11}^{-1}x + \alpha$. Similarly, $z_i^+(t)$ is the segment of $z^+(t)$ that initiates from the line $t = \lambda^{-1}_{1,i-1} x + \alpha$ and terminates on the line $t = \lambda_{1i}^{-1}x + \alpha$. The observation along path segment $z_i^+(t)$ is in the form

$$y_i^+(t) = H_i^+ u_{z_i}^+(t) \tag{12}$$

where H_i^+ is a constant $(m_i \times n)$ matrix.

Theorem 3: A necessary and sufficient condition for the observability of (11) and (12) in the region $R_\alpha(1)$ is that for each $(m_i \times n)$ matrix H_i^+ a new $(n - i + 1 \times n)$ matrix P_i^+

$$P_i^+ = \begin{bmatrix} p_{i,1} & p_{i,2} & \cdots p_{i,n} \\ p_{i+1,1} & p_{i+1,2} & \cdots p_{i+1,n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ p_{n,1} & & p_{n,n} \end{bmatrix} \tag{13}$$

can be formed that has the properties $p_{j,k} \neq 0$ for all $j = k$ and $p_{j,k} = 0$ for $j > k$, and each row of P_i^+ is formed by some linear combination of rows of H_i^+.

Proof: Let the $(n - i + 1 \times m_i)$ matrix B_i represent the transformation such that $B_i H_i^+ = P_i^+$. Then there exists a $g_i^+ = B_i y_i^+(t)$ such that $g_i^+ = P_i^+ u_{z_i}^+(t)$. Let $v_i^+(t)$ be a vector formed from deleting the first $i - 1$ components of $u_{z_i}^+(t)$ and W_i be the matrix formed from deleting the first $i - 1$ columns (all zeros) of P_i^+. Then we can write $g_i^+ = W_i v_i^+$. Since W_i will be nonsingular, $v_i^+ = W_i^{-1} g_i^+$. This implies that $u_i, \cdots, u_n$ can be recovered along $z_i^+(t)$ and therefore u_i can be recovered along $z_1^+(t), z_2^+(t), \cdots, z_i^+(t)$. Hence, u_i can be recovered along every characteristic line of type i in $R_\alpha(i)$ and the system is observable in $R_\alpha(1)$, proving sufficiency. Suppose for a particular H_i^+ a new matrix P_i^+ having the properties of Theorem 3 cannot be formed. This implies that some $u_j (i \leq j \leq n)$ cannot be recovered along $z_i^+(t)$. Hence, u_j cannot be recovered along some characteristic lines of type j in the region $R_\alpha(j)$, proving necessity.

Minimization of Performance Sensitivity for Time-Lag Systems

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Abstract—The problem of minimizing performance sensitivity to parameter variations in a nonlinear system containing time lag is studied. Only the scalar case is considered, but the approach is equally suitable to vector equations. An interesting special case is when the time lag is known imprecisely.

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Consider the scalar system represented by the equation

$$\dot{x}(t) = f[x(t), x(t - \tau), u(t), q, t] \tag{1}$$

where q is a parameter. It is desired to operate this system such that the following integral is minimized:

$$J = \int_0^T g(x, u, t)\, dt, \qquad \text{for fixed } T. \tag{2}$$

The usual procedure adopted is the direct application of the minimum principle [1] to obtain a set of necessary conditions. These necessary conditions then contain the assumed value of the parameter q. Assuming that the problem is well posed, the solution of these conditions will then give the minimum of the functional J corresponding to the assumed value for q. However, this optimal solution may be significantly sensitive to changes in q; in other words, if the actual q corresponding to the real plant were different from the value used in the optimality calculations, then the calculated optimal control could be a poor control to use on the physical plant. This problem has been investigated for the case without time lag in [2]. For a more extensive bibliography and some comments on the performance function modification approach to be proposed later in this correspondence, see [3].

The purpose of this correspondence is to obtain a control that, in some sense, is insensitive to parameter variations.

Let q^* be the best estimate of q. Now rewrite (1) indicating the dependence of the solution on q as follows:

$$\dot{x}(t, q) = f\{x(t, q), x(t - \tau, q), u(t), q, t\}. \tag{3}$$

Suppose that $q = q^* + \Delta q$, where Δq is a small variation. Expanding (3) by Taylor's theorem we get, to first order in Δq,

$$\frac{d}{dt}\left(\frac{dx}{dq}\Delta q\right) = \left(\frac{\partial f}{\partial x}\right)\frac{dx}{dq}\Delta q + \left(\frac{\partial f}{\partial x_\tau}\right)\frac{dx_\tau}{dq}\Delta q + \frac{\partial f}{\partial q}\Delta q \tag{4}$$

where x_τ denotes $x(t - \tau)$, and all derivatives are evaluated for q^*. Dividing (4) by Δq gives

$$\dot{y}(t) = \left(\frac{\partial f}{\partial x}\right) y(t) + \left(\frac{\partial f}{\partial x_\tau}\right) y(t - \tau) + \frac{\partial f}{\partial q} \tag{5}$$

where $y(t) \equiv (dx(t)/dq)$, the sensitivity coefficient.

Define a modified criterion function as follows:

$$\hat{J} = \int_0^T \{g(x, u, t) + W(t)y^2\}\, dt. \tag{6}$$

The choice of this criterion function is suggested by the following two requirements: 1) the derivation of a control function that produces "small" values for the functional defined by (2), and at the same time, 2) a control function that produces a trajectory insensitive, in some sense, to small variations in the parameter.

It is clear that the choice of weighting function $W(t)$ is of fundamental importance in this type of argument; the most suitable $W(t)$ is probably best found by numerical experimentation. Using the minimum principle [1], we now form

$$H = g(x, u, t) + Wy^2 + \lambda(t)f(x, x_\tau, u, q^*, t) + \mu(t)\left\{\left(\frac{\partial f}{\partial x}\right)y + \left(\frac{\partial f}{\partial x_\tau}\right)y_\tau + \frac{\partial f}{\partial q}\right\}. \tag{7}$$

The adjoint variables $\lambda(t)$, $\mu(t)$ must satisfy the following conditions:

$$\dot{\lambda}(t) = -\left[\frac{\partial g}{\partial x} + \lambda\frac{\partial f}{\partial x} + \mu\left\{\frac{\partial^2 f}{\partial x^2}y + \frac{\partial^2 f}{\partial x \partial x_\tau}y_\tau + \frac{\partial^2 f}{\partial q \partial x}\right\}\right]_t - \left[\lambda\frac{\partial f}{\partial x_\tau} + \mu\left\{\frac{\partial^2 f}{\partial x_\tau \partial x}y + \frac{\partial^2 f}{\partial x_\tau^2}y_\tau + \frac{\partial^2 f}{\partial q \partial x_\tau}\right\}\right]_{t+\tau}, \quad \text{on } [0, T - \tau] \tag{8}$$